Explicit Rational Solution of the KZ Equation (example)

Andrey Tydnyuk

E-mail address:andrey.tydniouk@verizon.net
735 Crawford Ave.Brooklyn, NY 11223. USA.

Abstract

We investigate the Knizhnik-Zamolodchikov linear differential system. The coefficients of this system are rational functions. We have proved that the solution of the KZ system is rational when k is equal to two and n is equal to three (see [5]) . In this paper, we construct the corresponding solution in the explicit form.
Mathematics Subject Classification (2000). Primary 34M05, Secondary 34M55,47B38.
Keywords: Symmetric group, linear differential system, rational solution.

Introduction

We will consider the system of the form:

$$\frac{dW}{dz} = -2A(z)W, \tag{0.1}$$

where $A(z)$ and $W(z)$ are $3{\times}3$ matrices, $z_1{\neq}z_2$. We suppose that $A(z)$ has the form

$$A(z) = \frac{P_1}{z - z_1} + \frac{P_2}{z - z_2}. \tag{0.2}$$

Here:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.3}$$

$$P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{0.4}$$

The matrices P_1 and P_2 are connected with the matrix representation of the symmetric group. System (0.1) is a special case of the Knizhnik-Zamolodchikov [1] , [2]. We have proved that the solution of system (0.1) is rational [5]. In this paper, we construct the corresponding solution in the explicit form. We consider the case when S_3 and use the method of L. Sakhnovich [3].

1 Main Notions, The Coefficients of the solution in the neighborhood of $z = \infty$

The solution $W(z)$ of system (0.1) has the form [5]:

$$W(z) = \frac{L_1}{(z - z_1)^2} + \frac{L_2}{(z - z_1)} + \frac{L_3}{(z - z_2)^2} + \frac{L_4}{(z - z_2)} + z^2 G_{-2} + zG_{-1} + G_0. \tag{1.1}$$

In a neighborhood of $z = \infty$ the solution $W(z)$ can be represented in the form

$$W(z) = \sum_{k=-2}^{\infty} z^{-k} G_k, \tag{1.2}$$

where the coefficients G_k are defined by the relations (see [3]).

$$[(q+1)I_3 - 2T]G_{q+1} = 2\sum_{r+s=q} T_r G_s, \quad r\geq 0 \tag{1.3}$$

and

$$T_r = z_1^{r+1}P_1 + z_2^{r+1}P_2, \quad T = P_1 + P_2. \tag{1.4}$$

The eigenvalues of T are

$$\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1. \tag{1.5}$$

The corresponding eigenvectors have the forms:

$$\ell_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \ell_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \ell_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}. \tag{1.6}$$

First, we will begin with finding all of the coefficients from G_{-2} to G_{-4}.
The eigenvalues of matrix $2T$ are twice the eigenvalues of the matrix T. Thus we get:

$$\mu_1 = 4, \quad \mu_2 = 2, \quad \mu_3 = -2. \tag{1.7}$$

The eigenvectors remain unchanged.
The smallest eigenvalue of $2T$ is equal to (-2). That is why we begin with the coefficient G_{-2}. From equation (1.3) we can say that

$$(-2I_3 - 2T)G_{-2} = 0. \tag{1.8}$$

Using equation (1.6) and (1.8) we conclude that

$$G_{-2} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \tag{1.9}$$

When the coefficient is G_{-1}, equation (1.3) takes the form

$$(-I_3 - 2T)G_{-1} = 2T_0G_{-2} \tag{1.10}$$

From the last relation we find that

$$G_{-1} = 2\begin{bmatrix} -(z_1+z_2) \\ z_2 \\ z_1 \end{bmatrix}. \tag{1.11}$$

When $q+1=0$, we get the relation:

$$-2TG_0 = 2(T_0G_{-1} + T_1G_{-2}). \tag{1.12}$$

From this we find that

$$G_0 = \begin{bmatrix} -z_1^2 + 4z_1z_2 - z_2^2 \\ z_1(z_1 - 2z_2) \\ z_2(-2z_1 + z_2) \end{bmatrix}. \tag{1.13}$$

When $q+1=1$ we obtain:

$$(I_3 - 2T)G_1 = 2(T_0G_0 + T_1G_{-1} + T_2G_{-2}). \tag{1.14}$$

Now we have

$$G_1 = 2\begin{bmatrix} 0 \\ (z_1 - z_2)^3 \\ -(z_1 - z_2)^3 \end{bmatrix}. \tag{1.15}$$

When $q+1=2$:

$$(2I_3 - 2T)G_2 = 2(T_0G_1 + T_1G_0 + T_2G_{-1} + T_3G_{-2}). \tag{1.16}$$

Remark 1.1
When $q+1=-1,0,1$ the matrices $(q+1)I_3 - 2T$ are invertible. That is why G_{-1},G_0,and G_1 are correctly defined by formulas (1.11),(1.13), and (1.15).
The situation changes when $q+1=2$ because 2 is an eigenvalue of the matrix $2T$. In this case, the matrix $2I_3 - 2T$ is not invertible.
The right-hand side of equation (1.16) has the form:

$$\begin{bmatrix} 4(z_1 - z_2)^4 \\ -2(z_1 - z_2)^4 \\ -2(z_1 - z_2)^4 \end{bmatrix}. \tag{1.17}$$

The eigenvalues of $(2I_3 - 2T)$ are

$$\mu_1 = -2, \quad \mu_2 = 0, \quad \mu_3 = 4. \tag{1.18}$$

The right side of (1.16) is the linear combination of the vectors ℓ_1 and ℓ_3.
From relations (1.6), (1.16), and (1.18) we obtain

$$G_2 = \begin{bmatrix} (z_1 - z_2)^4 \\ -\frac{1}{2}(z_1 - z_2)^4 \\ -\frac{1}{2}(z_1 - z_2)^4 \end{bmatrix}. \tag{1.19}$$

When $q+1=3$ we obtain:

$$(3I_3 - 2T)G_3 = 2(T_0G_2 + +T_1G_1 + T_2G_0 + T_3G_{-1} + T_4G_{-2}). \tag{1.20}$$

Using our previous results we find that

$$G_3 = \begin{bmatrix} \frac{3}{5}(z_1 - z_2)^4(z_1 + z_2) \\ \frac{1}{5}(z_1 - z_2)^3(6z_1^2 - 25z_1z_2 + 9z_2^2) \\ -\frac{1}{5}(z_1 - z_2)^3(9z_1^2 - 25z_1z_2 + 6z_2^2) \end{bmatrix}. \tag{1.21}$$

When $q+1=4$ we use the formula

$$(4I_3 - 2T)G_4 = 2(T_0G_3 + +T_1G_2 + T_2G_1 + T_3G_0 + T_4G_{-1} + T_5G_{-2}). \tag{1.22}$$

The right side of (1.22) has the form

$$\begin{bmatrix} \frac{9}{5}(z_1 - z_2)^4(3z_1^2 - 4z_1z_2 + 3z_2^2) \\ \frac{1}{5}(z_1 - z_2)^3(6z_1^3 - 8z_1^2z_2 - 71z_1z_2^2 + 33z_2^3) \\ -\frac{1}{5}(z_1 - z_2)^3(33z_1^3 - 71z_1^2z_2 - 8z_1z_2^2 + 6z_2^3) \end{bmatrix}. \tag{1.23}$$

The case when $q+1=4$ is similar to the case when $q+1=2$ (see Remark 1.1). The eigenvalues of $(4I_3 - 2T)$ are

$$\mu_1 = 0, \quad \mu_2 = 2, \quad \mu_3 = 6. \tag{1.24}$$

The right side of (1.22) is the linear combination of the vectors ℓ_2 and ℓ_3. From relations (1.22), (1.23), and (1.24) we obtain

$$G_4 = \begin{bmatrix} \frac{3}{10}(z_1 - z_2)^4(3z_1^2 - 4z_1z_2 + 3z_2^2) \\ \frac{1}{10}(z_1 - z_2)^3(15z_1^3 - 29z_1^2z_2 - 50z_1z_2^2 + 24z_2^3) \\ -\frac{1}{10}(z_1 - z_2)^3(24z_1^3 - 50z_1^2z_2 - 29z_1z_2^2 + 15z_2^3) \end{bmatrix}. \tag{1.25}$$

From (1.1) we wind up with the following system:

$$L_2 + L_4 = G_1 \tag{1.26}$$

$$L_1 + L_2z_1 + L_3 + L_4z_2 = G_2 \tag{1.27}$$

$$2L_1z_1 + L_2z_1^2 + 2L_3z_2 + L_4z_2^2 = G_3 \tag{1.28}$$

$$3L_1z_1^2 + L_2z_1^3 + 3L_3z_2^2 + L_4z_2^3 = G_4 \tag{1.29}$$

System (1.26)-(1.29) can be written in the matrix form:

$$SX = Y, \tag{1.30}$$

where

$$S = \begin{bmatrix} 0 & I_3 & 0 & I_3 \\ I_3 & z_1 & I_3 & z_2 \\ 2z_1 I_3 & z_1^2 I_3 & 2z_2 I_3 & z_2^2 I_3 \\ 3z_1^2 I_3 & z_1^3 I_3 & 3z_2^2 I_3 & z_2^3 I_3 \end{bmatrix}, \tag{1.31}$$

$$X = \mathrm{col}[L_1, L_2, L_3, L_4], \tag{1.32}$$

$$Y = \mathrm{col}[G_1, G_2, G_3, G_4]. \tag{1.33}$$

In equation (1.30) the matrices S and Y are known, but the matrix X is unknown.

From relation (1.30) we get that

$$X = S^{-1}Y \tag{1.34}$$

where

$$S^{-1} = \begin{bmatrix} -\frac{z_1 z_2^2}{(z_1-z_2)^2} I_3 & \frac{z_2(2z_1+z_2)}{(z_1-z_2)^2} I_3 & -\frac{z_1+2z_2}{(z_1-z_2)^2} I_3 & \frac{1}{(z_1-z_2)^2} I_3 \\ \frac{(3z_1-z_2)z_2^2}{(z_1-z_2)^3} I_3 & -\frac{6z_1 z_2}{(z_1-z_2)^3} I_3 & \frac{3z_1+z_2}{(z_1-z_2)^3} I_3 & \frac{2}{(-z_1+z_2)^3} I_3 \\ -\frac{z_1^2 z_2}{(z_1-z_2)^2} I_3 & \frac{z_1(z_1+2z_2)}{(z_1-z_2)^2} I_3 & -\frac{2z_1+z_2}{(z_1-z_2)^2} I_3 & \frac{1}{(z_1-z_2)^2} I_3 \\ \frac{z_1^2(z_1-3z_2)}{(z_1-z_2)^3} I_3 & \frac{6z_1 z_2}{(z_1-z_2)^3} I_3 & -\frac{3z_1+z_2}{(z_1-z_2)^3} I_3 & \frac{2}{(z_1-z_2)^3} I_3 \end{bmatrix}. \tag{1.35}$$

Thus, we find that

$$L_1 = \begin{bmatrix} \frac{1}{10}(3z_1 - 7z_2)(z_1 - z_2)^3 \\ \frac{1}{10}(3z_1 - 7z_2)(z_1 - z_2)^3 \\ -\frac{1}{5}(3z_1 - 7z_2)(z_1 - z_2)^3 \end{bmatrix}, \tag{1.36}$$

$$L_2 = \begin{bmatrix} 0 \\ \frac{1}{5}(3z_1 - 7z_2)(z_1 - z_2)^2 \\ -\frac{1}{5}(3z_1 - 7z_2)(z_1 - z_2)^2 \end{bmatrix}, \tag{1.37}$$

$$L_3 = \begin{bmatrix} \frac{1}{10}(7z_1 - 3z_2)(z_1 - z_2)^3 \\ -\frac{1}{5}(7z_1 - 3z_2)(z_1 - z_2)^3 \\ \frac{1}{10}(7z_1 - 3z_2)(z_1 - z_2)^3 \end{bmatrix}, \tag{1.38}$$

and

$$L_4 = \begin{bmatrix} 0 \\ \frac{1}{5}(7z_1 - 3z_2)(z_1 - z_2)^3 \\ -\frac{1}{5}(7z_1 - 3z_2)(z_1 - z_2)^3 \end{bmatrix}. \tag{1.39}$$

This way we have proved the following statement:

Proposition 1 *System (0.1) has the following solution:*

$$W_1(z) = \frac{L_1}{(z-z_1)^2} + \frac{L_2}{(z-z_1)} + \frac{L_3}{(z-z_2)^2} + \frac{L_4}{(z-z_2)} + z^2G_{-2} + zG_{-1} + G_0. \tag{1.40}$$

The matrices G_k and L_k are defined by the relations (1.9), (1.1), (1.13), and (1.36) - (1.39).

To find the next solution to the system (0.1) we consider the case

$$g_k = 0 \quad when \quad k < 2 \tag{1.41}$$

In this case, we have

$$g_2 = \mathrm{col}[0, 1, -1]. \tag{1.42}$$

From relation (1.3) we get:

$$(3I_3 - 2T)g_3 = T_0 g_2. \tag{1.43}$$

From this we find that

$$g_3 = \frac{1}{5}\begin{bmatrix} z_1 - z_2 \\ 2z_1 + 3z_2 \\ -3z_1 - 2z_2 \end{bmatrix}. \tag{1.44}$$

In order to find g_4 we will use equation (1.3) again;

$$(4I_3 - 2T)g_4 = T_0 g_3 + T_1 g_2. \tag{1.45}$$

The right side of the equation (1.45) has the form:

$$\frac{1}{5}\begin{bmatrix} 7(z_1 - z_2)(z_1 + z_2) \\ z_1^2 + z_1 z_2 + 8z_2^2 \\ -8z_1^2 - z_1 z_2 - z_2^2 \end{bmatrix} = \begin{bmatrix} \phi \\ -\frac{\phi}{2} \\ -\frac{\phi}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ \psi + \frac{\phi}{2} \\ -\psi - \frac{\phi}{2} \end{bmatrix}, \tag{1.46}$$

where

$$\phi = \frac{7}{5}(z_1 - z_2)(z_1 + z_2) \quad , \quad \psi = \frac{1}{5}z_1^2 + z_1 z_2 + 8z_2^2 \tag{1.47}$$

Analogously (1.34) we can write:

$$X = S^{-1}Y, \tag{1.48}$$

where

$$X = \mathrm{col}[M_1, M_2, M_3, M_4], \tag{1.49}$$

$$Y = \mathrm{col}[0, g_2, g_3, g_4]. \tag{1.50}$$

Thus, we can say that

$$M_1 = \left[\begin{array}{c} \frac{5z_1+3z_2}{10(z_1-z_2)} \\ \frac{z_1z_2+9(-2z_1^2+2z_1z_2+z_2^2)}{30(z_1-z_2)^2} \\ -\frac{z_1z_2-3z_1(z_1-4z_2)}{30(z_1-z_2)^2} \end{array} \right], \tag{1.51}$$

$$M_2 = \left[\begin{array}{c} -\frac{4(z_1+z_2)}{5(z_1-z_2)^2} \\ -\frac{-24z_1^2+46z_1z_2-12z_2^2}{15(z_1-z_2)^3} \\ \frac{-12z_1^2++46z_1z_2-24z_2^2}{15(z_1-z_2)^3} \end{array} \right], \tag{1.52}$$

$$M_3 = \left[\begin{array}{c} -\frac{3z_1+5z_2}{10(z_1-z_2)} \\ \frac{z_1z_2+3(4z_1-z_2)z_2}{30(z_1-z_2)^2} \\ -\frac{z_1z_2+9(z_1^2+2z_1z_2-2z_2^2}{30(z_1-z_2)^2} \end{array} \right], \tag{1.53}$$

and

$$M_4 = \left[\begin{array}{c} \frac{4(z_1+z_2)}{5(z_1-z_2)^2} \\ \frac{-24z_1^2+46z_1z_2-12z_2^2}{15(z_1-z_2)^3} \\ -\frac{-12z_1^2+46z_1z_2-24z_2^2}{15(z_1-z_2)^3} \end{array} \right]. \tag{1.54}$$

This way we have proved the following statement:

Proposition 2 *System (0.1) has the following solution:*

$$W_2(z) = \frac{M_1}{(z-z_1)^2} + \frac{M_2}{(z-z_1)} + \frac{M_3}{(z-z_2)^2} + \frac{M_4}{(z-z_2)}. \tag{1.55}$$

The matrices M_k are defined by the relations (1.51) - (1.54).

The next solution of system (0.1) has the form

$$W_3(z) = \frac{N_1}{(z-z_1)^2} + \frac{N_2}{(z-z_1)} + \frac{N_3}{(z-z_2)^2} + \frac{N_4}{(z-z_2)}. \tag{1.56}$$

In order to find N_1 , N_2 , N_3 , and N_4 we consider the case when

$$G_k = 0 \quad when \quad k < 4 \quad and \quad G_4 = \ell_1. \tag{1.57}$$

From relations (1.34) and (1.35) we have

$$N_1 = N_3 = \frac{1}{(z_1 - Z_2)^2}\ell_1 \tag{1.58}$$

$$N_2 = -N_4 = \frac{2}{(-z_1+z_2)^3}. \tag{1.59}$$

The main theorem follows from Propositions 1 - 3.

Theorem 1

The general solution of system (0.1) has the form:

$$W_z = \alpha_1 W_1(z) + \alpha_2 W_2(z) + \alpha_3 W_3(z), \tag{1.60}$$

where α_1 , α_2 , and α_3 are arbitrary constants.

References

1. Chervov A., Talalaev D., KZ equation, G-opers, quantum Drinfeld-Sokolov reduction and quantum Cayley-Hamilton identity, arXiv:hep-th/0607250, 2006.
2. Etingof P.I., Frenkel I.B., Kirillov A.A. jr.,Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations, Amer. Math. Society, 1998.
3. Sakhnovich L.A., Meromorphic Solutions of Linear Differential Systems, Painleve Type Functions, Journal of Operator and Matrix Theory, v. 1, 87-111, 2007.
4. Sakhnovich L.A., Rational Solution of the KZ Equation, existence and construction, arXiv:math.ph/0609067, 2006.
5. Tydnyuk A.I., Rational Solution of the KZ Equation (example), arXiv:math.CA/0612153, 2006.